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# 2015 Spring - Information Theory

## Homework 2 (Due to Mar.20)

### Part 1: Evaluate the entropy of a distribution

Let  $K$  and  $Z$  to have a uniform distribution over the set  $\{1, 2, 3, 4\}$  and independent, let moreover

$$X = K + Z$$

$$Y = K - Z.$$

Now evaluate

- $H(X)$  and  $H(Y)$ ,
- $H(X|Y)$  and  $H(Y|X)$ ,
- $H(X, Y)$ ,  $H(X, Y|Z)$  and  $H(X, Y|K)$ ,
- $I(X; Y)$  and  $I(X; Y|K)$ .

### Part 3: Entropy of functionals

What happens to entropy of a function of a random variable?

What is the relationship between  $H(X)$  and  $H(Y)$  when

- $Y = 2^X$ ,
- $Y = \cos(X)$ ,
- $Y = \lfloor X^3 \rfloor$ .

what is in general the relationship between  $H(X)$  and  $H(Y)$  for  $Y = f(X)$  ?

### Part 3: Convexity properties of information measures

Prove the following inequalities involving convexity of information measures:

- $H(X)$  is concave in  $p_x(x)$ .
- $I(X; Y)$  is convex in  $p_{Y|X}(y|x)$  for a fixed  $P_X(x)$ .
- $I(X; Y)$  concave in  $P_X(x)$  for a fixed  $p_{Y|X}(y|x)$ .

### Part 4: Inequalities for mutual information

Given 3 random variables, what is the relationship between  $I(X; Y)$  and  $I(X; Y|Z)$  ?

Under what conditions we have that  $I(X; Y) \geq I(X; Y|Z)$  ?

### Part 5: Alternative proof of the positivity of the relative entropy

Provide an alternative proof of the positivity of the relative entropy  $D(p||q)$

- once using the log-sum inequality,
- once using the Jensen inequality.

When possible, show that the inequality holds with equality when  $p = q$ .

### Part 6: Prefix Codes

Prove lemma 3.5 in the notes.

### Part 7: Suffix Codes

We have shown few properties of prefix codes. What of those properties still hold for suffix codes, that is codes in which no codeword is a suffix of another codeword?

### Part 8: Matlab Exercise

Graphs are very important in many applications: build a game tree for a tic-tac-toe game where each node corresponds to a move by a first player and a move by the second player.

Then determine

- which paths lead to victory? which one to a draw?
- what is the average depth of the tree?
- what is the shortest game? what is the longest?
- Given that you are the first player, what are your chances to win the game?
- Given that you are the second player, what are your chances to win the game?