Analog to digital conversion

Sampling and reconstruction

In the past lecture

- Analog to digital conversion
- Sampling of Signals and Signal
- Quantization

In the past lecture

- Why do we want to have transmissions in the digital domain?
- What does the conversion between analog and digital entails?
- Describe the sampling theorem?

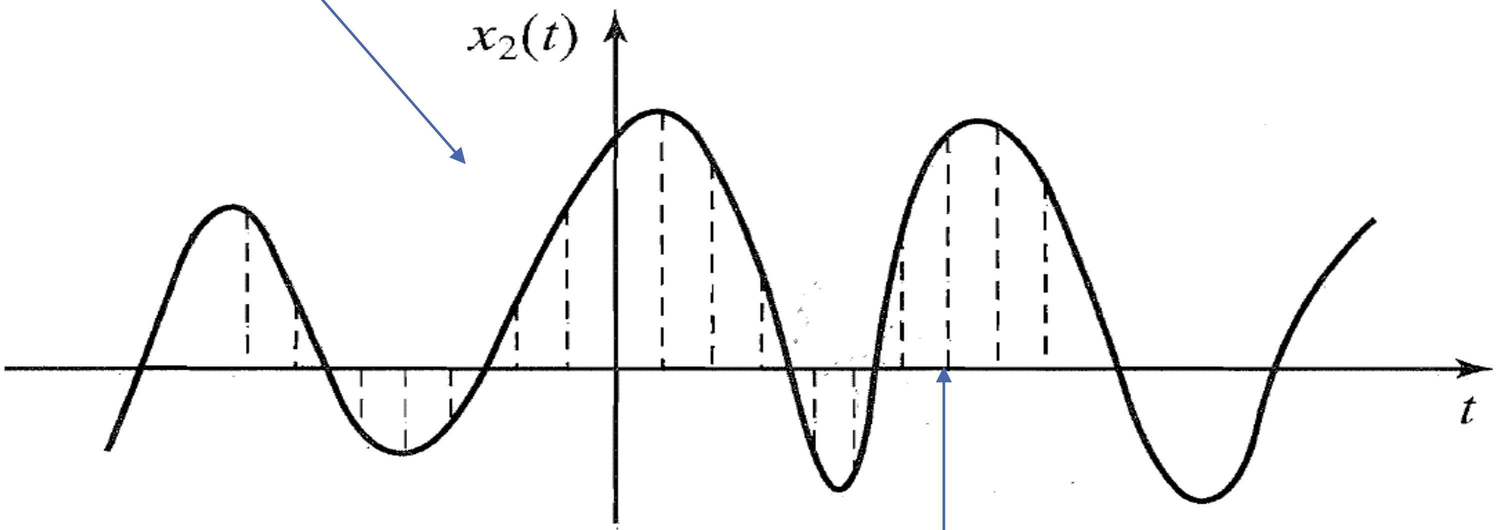
Let’s continue from last time

- We consider an analog source
  - Continuous time and continuous values
- Sample
  - Turn continuous time into discrete time
- Quantize
  - Turn continuous amplitude into discrete amplitudes
- Transmit over the channel
- Reconstruct at the destination

Sampling

- **C2D**
  We take equally spaced samples of the analog signal

Continuous time signal

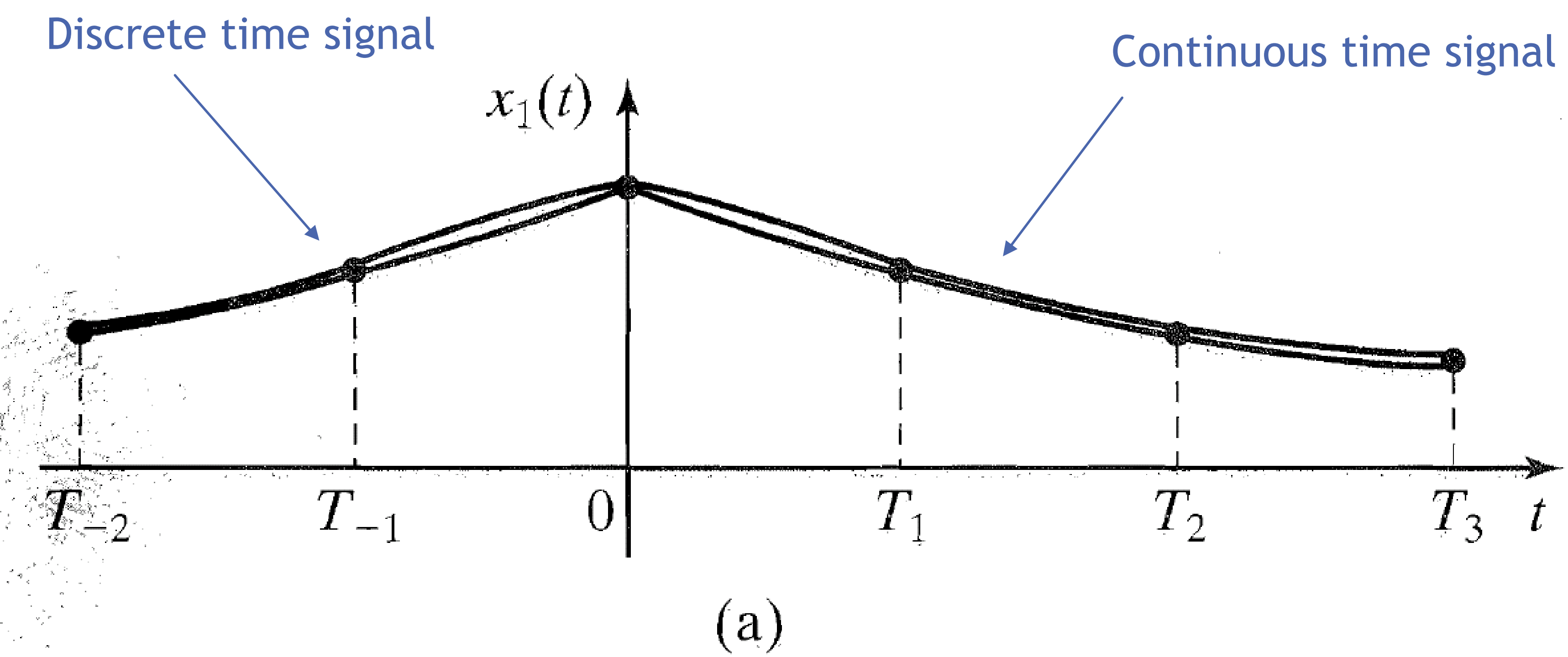


(b)

Discrete time signal

Sampling

- **D2C**
  We reconstruct the signal from the samples through some form of (maybe random) interpolation



Sampling

The idea leading to the sampling theorem is very simple and quite intuitive. Let us assume that we have two signals, $x_1(t)$ and $x_2(t)$, as shown in Figure 7.1. The first signal $x_1(t)$ is a smooth signal, it varies very slowly; therefore, its main frequency content is at low frequencies. In contrast, $x_2(t)$ is a signal with rapid changes due to the presence of high-frequency components. We will approximate these signals with samples taken at regular intervals T_1 and T_2, respectively. To obtain an approximation of the original signal, we can use linear interpolation of the sampled values. It is obvious that the sampling interval for the signal $x_1(t)$ can be much larger than the sampling interval necessary to reconstruct signal $x_2(t)$ with comparable distortion.

Sampling

- We know that using linear interpolation is not really very efficient
- We have already developed a theory to represent a signals over sinusoidal basis
  - We can use this approach instead!

Sampling Theorem

Sampling Theorem. Let the signal $x(t)$ have a bandwidth W, i.e., let $X(f) \equiv 0$ for $|f| \geq W$. Let $x(t)$ be sampled at multiples of some basic sampling interval T_s, where $T_s \leq \frac{1}{2W}$, to yield the sequence $\{x(nT_s)\}_{n=-\infty}^{+\infty}$. Then it is possible to reconstruct the original signal $x(t)$ from the sampled values by the reconstruction formula

$$x(t) = \sum_{n=-\infty}^{\infty} 2W'T_s x(nT_s) \operatorname{sinc}[2W'(t - nT_s)], \tag{7.1.1}$$

where W' is any arbitrary number that satisfies the condition

$$W \leq W' \leq \frac{1}{T_s} - W.$$

Sampling Theorem

Proof. Let $x_\delta(t)$ denote the result of sampling the original signal by impulses at nT_s time instants. Then

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s). \qquad (7.1.2)$$

We can write

$$x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \qquad (7.1.3)$$

Sampling Theorem

where we have used the property that $x(t)\delta(t - nT_s) = x(nT_s)\delta(t - nT_s)$. Now if we find the Fourier transform of both sides of the preceding relation and apply the dual of the convolution theorem to the right-hand side, we obtain

$$X_\delta(f) = X(f) \star \mathscr{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]. \qquad (7.1.4)$$

Sampling Theorem

Using Table 2.1 to find the Fourier transform of $\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$, we obtain

$$\mathscr{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right). \tag{7.1.5}$$

Substituting Equation (7.1.5) into Equation (7.1.4), we obtain

$$\begin{aligned} X_\delta(f) &= X(f) \star \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right) \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T_s}\right), \end{aligned} \tag{7.1.6}$$

Sampling Theorem

For $W \leq |f| < \frac{1}{T_s} - W$, the filter can have any characteristics that makes its implementation easy. Of course, one obvious (though not practical) choice is an ideal lowpass filter with bandwidth W', where W' satisfies $W \leq W' < \frac{1}{T_s} - W$, i.e., by using a filter with a transfer function given by

$$H(f) = T_s \prod\left(\frac{f}{2W'}\right). \tag{7.1.7}$$

Sampling Theorem

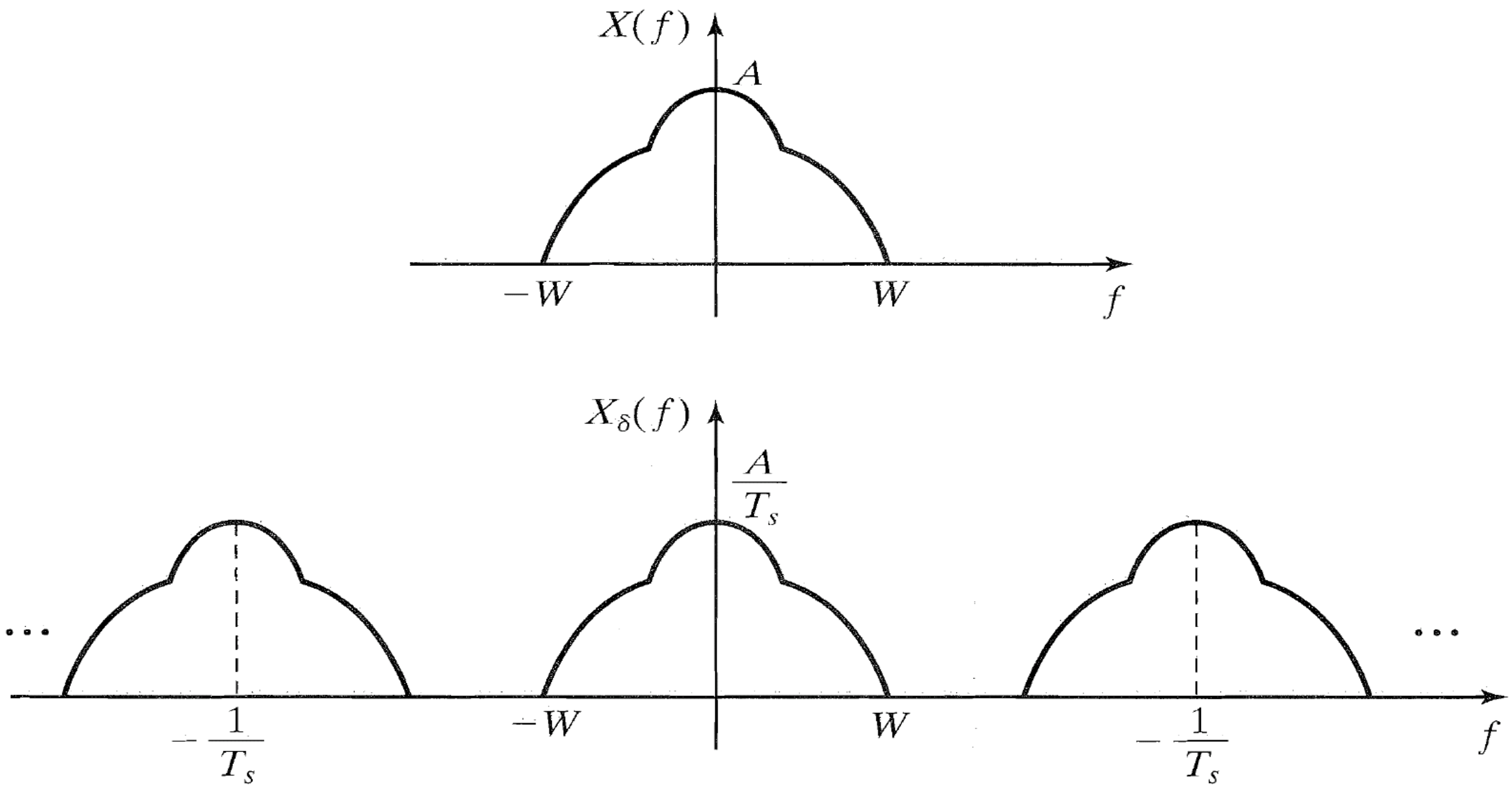


Figure 7.2 Frequency-domain representation of the sampled signal.

Sampling Theorem

With this choice, we have

$$X(f) = X_\delta(f) T_s \prod\left(\frac{f}{2W'}\right). \qquad (7.1.8)$$

Taking the inverse Fourier transform of both sides, we obtain

$$\begin{aligned} x(t) &= x_\delta(t) \star 2W' T_s \operatorname{sinc}(2W' t) \\ &= \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)\right) \star 2W' T_s \operatorname{sinc}(2W' t) \\ &= \sum_{n=-\infty}^{\infty} 2W' T_s x(nT_s) \operatorname{sinc}\left(2W'(t - nT_s)\right). \end{aligned} \qquad (7.1.9)$$

Sampling Theorem

This relation shows that if we use sinc functions for interpolation of the sampled values, we can perfectly reconstruct the original signal. The sampling rate $f_s = \frac{1}{2W}$ is the minimum sampling rate at which no aliasing occurs. This sampling rate is known as the *Nyquist sampling rate*. If sampling is done at the Nyquist rate, then the only choice for the reconstruction filter is an ideal lowpass filter and $W' = W = \frac{1}{2T_s}$. In this case,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \operatorname{sinc}(2Wt - n)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}\left(\frac{t}{T_s} - n\right). \quad (7.1.10)$$

Sampling Theorem: an example

Example 7.1.1

In this development, we have assumed that samples are taken at multiples of T_s. What happens if we sample regularly with T_s as the sampling interval, but the first sample is taken at some $0 < t_0 < T_s$?

Sampling Theorem: an example

Solution We define a new signal $y(t) = x(t + t_0)$. Then $y(t)$ is bandlimited with $Y(f) = e^{j2\pi f t_0} X(f)$ and the samples of $y(t)$ at $\{kT_s\}_{k=-\infty}^{\infty}$ are equal to the samples of $x(t)$ at $\{t_0 + kT_s\}_{k=-\infty}^{\infty}$. Applying the sampling theorem to the reconstruction of $y(t)$, we have

$$y(t) = \sum_{k=-\infty}^{\infty} y(kT_s) \operatorname{sinc}(2W(t - kT_s))$$

$$= \sum_{k=-\infty}^{\infty} x(t_0 + kT_s) \operatorname{sinc}(2W(t - kT_s));$$

hence,

$$x(t + t_0) = \sum_{k=-\infty}^{\infty} x(t_0 + kT_s) \operatorname{sinc}(2W(t - kT_s)).$$

Substituting $t = -t_0$, we obtain the following important interpolation relation:

$$x(0) = \sum_{k=-\infty}^{\infty} x(t_0 + kT_s) \operatorname{sinc}(2W(t_0 + kT_s)). \quad (7.1.11)$$

Sampling Theorem: an example

Example 7.1.2

A bandlimited signal has a bandwidth equal to 3400 Hz. What sampling rate should be used to guarantee a guard band of 1200 Hz?

Sampling Theorem: an example

Example 7.1.2

A bandlimited signal has a bandwidth equal to 3400 Hz. What sampling rate should be used to guarantee a guard band of 1200 Hz?

Solution We have

$$f_s = 2W + W_G;$$

therefore,

$$f_s = 2 \times 3400 + 1200 = 8000. \qquad \blacksquare$$

Time for a break

See you in 15mins