2015 Fall - Information Theory

Homework 5 (Due on Dec. 7th)

Part 1: Alternative representation of a discrete channel

Let X and Y be respectively the input and the output of a discrete channel $p(y|x)$. Show that there exists a random variable Z and a function f such that $Y = f(X, Z)$ and Z is independent of X. This is a generalization of the alternative representation of a BSC at the attachment.

Let X_i and Y_i be respectively the input and the output of the DMC $p(y|x)$. Then Y_i depends causally on X_i via $Y_i = f(X_i, Z_i)$, where Z_i is the noise random variable at time i which has the same distribution as Z, and Z_i is independent of all the random variables which have already been generated in the system.

Part 2: Channel capacity

Derive the capacity of the parallel of two channels and the product of two channels. C_1 with transition probability P_1, C_2 with transition probability P_2,

- product channel $C_{\text{product}} \leq \min\{C_1, C_2\}$
- parallel channel $C_{\text{parallel}} = C_1 + C_2$

Part 3:Order Statistics

Suppose that we have a basic random experiment, and X is a real-valued random variable for the experiment with CDF F and PDF f. We perform n independent replications of the basic experiment to generate a random sample $X = (X_1, X_2, ..., X_n)$ of size n from the distribution of X. Recall that this is a sequence of independent random variables, each with the distribution of X. Let $X_n, k(X)$ denote the kth smallest element of the sample X. This statistics is called the order statistic of order k. Usually, the first step in a statistical study is to order the data; thus order statistics occur naturally. Our goal in this section is to study the distribution of the order statistics in terms of the sampling distribution. Note in particular that the extreme order statistics are the minimum and maximum values: $X_{n,1} = \min(X_1, X_2, ..., X_n), X_{n,n} = \max(X_1, X_2, ..., X_n)$

Let $G_{n,k}$ denote the CDF of $X_{n,k}$. Define $N_{n,y} = \#(\{i \in 1, 2, ..., n : X_i \leq u\}), y \in \mathbb{R}$

- Show that $N_{n,y}$ has the binomial distribution with parameter n and $F(y)$ for each $y \in \mathbb{R}$
- Show that $X_{n,k} \leq y$ if and only if $N_{n,y} \geq k$ for $y \in \mathbb{R}$ and $k \in \{1, 2, 3, ..., n\}$
- Use the previous two results to show that $G_{n,k}(y) = \sum_{j=k}^{n} \binom{n}{j} F(y)^j (1 - F(y))^{n-j}, y \in \mathbb{R}$
- Show that $G_{n,1}(y) = 1 - (1 - F(y))^n, y \in \mathbb{R}$
- Show that $G_{n,n}(y) = F(y)^n, y \in \mathbb{R}$
- Suppose now that X has a continuous distribution. Show that $X_{n,k}$ has a continuous distribution with PDF $g_{n,k}(y) = \binom{n}{k-1,1,n-k} F(y)^{k-1} (1 - F(y))^{n-k} f(y), y \in \mathbb{R}$

Part 4: Matlab Exercise

According to section 9.8 of the textbook, please construct a random encoder, a MAP decoder and a threshold decoder for the BSC with finite codeword length. Plot the error probabilities of two decoders as a function of codeword length (n).