2015 Fall - Information Theory

Homework 2 (Due to Oct.9th)

Part 1: Evaluate the entropy of a distribution

Let K and Z to have a uniform distribution over the set $\{1, 2, 3, 4\}$ and independent, let moreover

$$X = K + Z$$
$$Y = K - Z.$$

Now evaluate

- $H(X)$ and $H(Y)$,
- $H(X|Y)$ and $H(Y|X)$,
- $H(X, Y)$, $H(X, Y|Z)$ and $H(X, Y|K)$,
- $I(X; Y)$ and $I(X; Y|K)$.

Part 3: Entropy of functionals

What happens to entropy of a function of a random variable?
What is the relationship between $H(X)$ and $H(Y)$ when

- $Y = 2^X$,
- $Y = \cos(X)$,
- $Y = [X^3\ X]$.

what is in general the relationship between $H(X)$ and $H(Y)$ for $Y = f(X)$?

Part 3: Convexity properties of information measures

Prove the following inequalities involving convexity of information measures:

- $H(X)$ is concave in $p_x(x)$.
- $I(X; Y)$ is convex in $p_{Y|X}(y|x)$ for a fixed $P_X(x)$.
- $I(X; Y)$ concave in $P_X(x)$ for a fixed $p_{Y|X}(y|x)$.

Part 4: Inequalities for mutual information

Given 3 random variables, what is the relationship between $I(X; Y)$ and $I(X; Y|Z)$?
Under what conditions we have that $I(X; Y) \geq I(X; Y|Z)$?

Part 5: Alternative proof of the positivity of the relative entropy

Provide and alternative proof of the positivity of the relative entropy $D(p||q)$

- once using the log-sum inequality,
- once using the Jensen inequality.

When possible, show that the inequality holds with equality when $p = q$.

Part 8: Matlab Exercise

Graphs are very important in many applications, so it is extremely important for you to learn how to represent and navigate a tree. This week you shall indeed learn just how to do so. In the attached Matlab file you shall find the code to help your program an automatic player of the game of tic-tac-toe in which:

- the state of the game is represented by a 3×3 ternary matrix: 0 means that the cell is not occupied, 1 means that is occupied by player one and 2 means that it is occupied by player two,
- all the possible 3^9 state of the games represents the nodes in a tree,
- the edges in the tree are represented by a $3^9 \times 3^9$ matrix: a 1 in positing $i \times j$ means that the node i is connected to the node j,
- write a function that identifies the states in which player 1 or player 2 win the game,
- write a function that identifies the state in which the state is a draw,
- write a function that, given an initial state, identifies the winning states for player 1, player 2 and the draws.

Note that:

- the tree is very large, so it's represented using a global variable
- the tree is very sparse (95% of the entries are zero) so its represented using a sparse variable.
- if you use any form of loop to navigate the tree, this will slow down your execution significantly, so absolutely avoid loops for the last function.