2015 Fall - Information Theory

Review Exercise for Stochastic Process (No need to hand in)

Part 1: Gambler's ruin

(From *Introduction to Probability Models 10th ed. by Sheldon M. Ross* example 4.30)
Consider the gambler's ruin problem with $p = 0.4$(winning prob) and $N = 7$(the maximum amount of units you can win). Starting with 3 units, determine

(a) find the transition matrix $\mathbf{P_T}$,
(b) the expected amount of time the gambler has 5 units,
(c) the expected amount of time the gambler has 2 units.

Solution:

$$\mathbf{P_T} = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1t} \\ P_{21} & P_{22} & \cdots & P_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ P_{t1} & P_{t2} & \cdots & P_{tt} \end{pmatrix}$$

The $\mathbf{P_T}$ specifies only the transition probs from transient states into transient states, some of it's row sums are less than 1.
Let s_{ij} denote the expected number of time periods that the Markov chain is in state j, given that it starts in state i.
Let $\delta_{ij} = 1$ when $i = j$ and let it be 0 otherwise.
$s_{ij} = \delta_{ij} + \sum_{k=1}^{t} P_{ik} s_{kj}$
Let $\mathbf{S}$ denote the matrix of values s_{ij}, $i, j = 1, \ldots, t$.

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1t} \\ s_{21} & s_{22} & \cdots & s_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ s_{t1} & s_{t2} & \cdots & s_{tt} \end{pmatrix}$$

$\mathbf{S} = \mathbf{I} + \mathbf{P_T}\mathbf{S}$
$(\mathbf{I} - \mathbf{P_T})\mathbf{S} = \mathbf{I}$
$\mathbf{S} = (\mathbf{I} - \mathbf{P_T})^{-\mathbf{1}}$

(a) $$\mathbf{P_T} = \begin{pmatrix} 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \end{pmatrix}$$

for (b) and (c)

$$\mathbf{S} = (\mathbf{I} - \mathbf{P_T})^{-\mathbf{1}} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1.4206 & 2.3677 & 2.9990 & 1.7533 & 0.9228 & 0.3691 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(b) $s_{3,5} = 0.9228$.
(c) $s_{3,2} = 2.3677$.

Part 2: Flipping coin

A fair coin is flipped till 3 consecutive heads are seen (then you stop flipping).

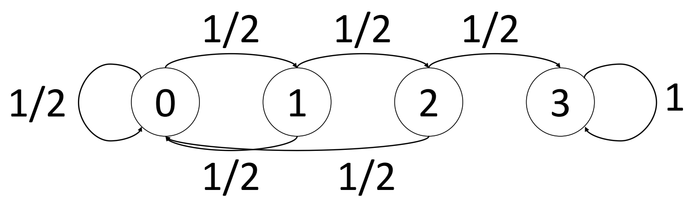
(a) Draw the transition probability diagram,

(b) Find the expected mean of flips.

Solution:

state $S = \{0, 1, 2, 3\}$

(a) Transition prob diagram:

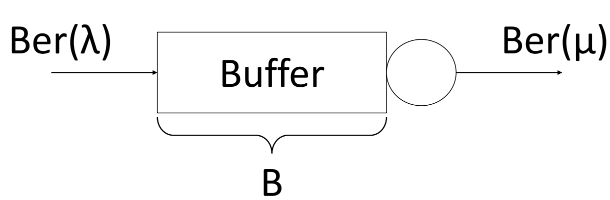


(b) Let T_{ij} denotes as expected time to hit state j starting from state i.

$T_{03} = \frac{1}{2}(1 + T_{03}) + \frac{1}{2}(1 + T_{13})$
$T_{13} = \frac{1}{2}(1 + T_{03}) + \frac{1}{2}(1 + T_{23})$
$T_{23} = \frac{1}{2}(1 + T_{03}) + \frac{1}{2}$
$\vdots$
$T_{03} = 14$

Part 3: Queueing with finite buffer length for DTMC

Assume packets go into the system with rate λ and leave the system with rate μ as the figure. Let the arrival happens before departures in each time slot. The buffer length is B.



(a) Draw the transition probability diagram,

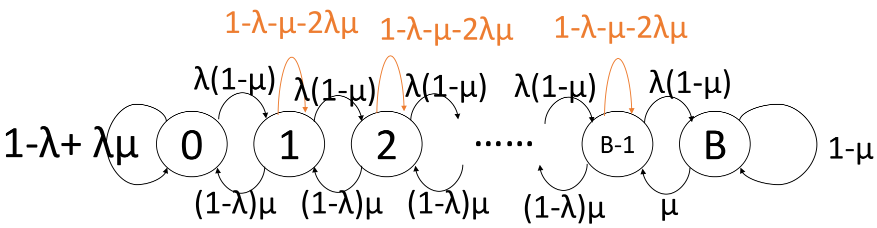
(b) Find the transition matrix,

(c) Find the stationary distribution in terms of λ, μ, and B.

Solution:

states $\mathbf{S} = \{0, 1, 2, \cdots, B-1, B\}$,

(a) transition prob diagram:



(b) transition matrix (the matrix here is a $(B+1) \times (B+1)$ matrix)

$$\mathbf{P_T} = \begin{pmatrix} 1-\lambda+\lambda\mu & \lambda(1-\mu) & 0 & 0 & \cdots & 0 & 0 \\ (1-\lambda)\mu & 1-\lambda-\mu-2\lambda\mu & \lambda(1-\mu) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & (1-\lambda)\mu & 1-\lambda-\mu-2\lambda\mu & \lambda(1-\mu) & 0 \\ 0 & \cdots & 0 & 0 & (1-\lambda)\mu & 1-\lambda-\mu-2\lambda\mu & \lambda(1-\mu) \\ 0 & \cdots & 0 & 0 & 0 & \mu & 1-\mu \end{pmatrix}$$

(c) A useful trick: If $\pi_j P_{ij} = \pi_j P_{ji} \ \forall j \neq i$, then we can have $\pi^T = \pi^T \cdot \mathbf{P_T}$ (if it satisfies the time reversible property)

$$\begin{aligned}
&\lambda(1-\mu)\pi_0 = (1-\lambda)\mu\pi_1 \\
&\lambda(1-\mu)\pi_1 = (1-\lambda)\mu\pi_2 \\
&\vdots \\
&\lambda(1-\mu)\pi_{B-2} = (1-\lambda)\mu\pi_{B-1} \\
&\lambda(1-\mu)\pi_{B-1} = \mu\pi_B \\
&\pi_{i+1} = \frac{\lambda(1-\mu)}{(1-\lambda)\mu}\pi_i, 0 \leq i \leq B-2 \\
&\pi_B = \frac{\lambda(1-\mu)}{\mu}\pi_{B-1} \\
&\pi_i = \left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^i \pi_0, i \leq B-1 \\
&\pi_B = \frac{\lambda(1-\mu)}{\mu}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{B-1} \pi_0 \\
&\text{Using } \sum_i \pi_i = 1 \Longrightarrow \pi_0 = \left(\sum_{i=1}^{B-1}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^i + \frac{\lambda(1-\mu)}{\mu}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{B-1}\right)^{-1}
\end{aligned}$$

Part 4: Queueing with infinite buffer length for DTMC(Guessing Theorem)

As the description as Part 3, but the buffer length B is infinite here.

(a) Under what condition the stationary distribution will exist? And what is the stationary distribution?
(b) Find the expected queue length in steady-state.

Solution:
From Part 3 we know $\pi_{i+1} = \frac{\lambda(1-\mu)}{(1-\lambda)\mu}\pi_i$, now we let $\rho = \frac{\lambda(1-\mu)}{(1-\lambda)\mu}$

$\Rightarrow \pi_i = \rho^i \pi_0, \sum_{i=0}^{\infty} \pi_i = 1$

$\Rightarrow \pi_0 \sum_{i=1}^{\infty} \rho^i = 1$

(a) if we want the stationary distribution exist, then we need $\sum_{i=0}^{\infty} \rho^i$ to be finite $\Rightarrow \rho < 1$.

$\pi_0 \frac{1}{1-\rho} = 1 \Rightarrow \pi_0 = (1-\rho)$

$\pi_i = (1-\rho)\rho^i, i = 0, 1, \cdots$

(b) E(queue length in steady-state)

$= \sum_{i=0}^{\infty} i\pi_i$

$= \sum_{i=0}^{\infty} i\rho^i(1-\rho)$

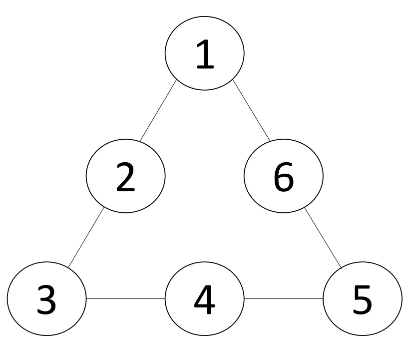
$= (1-\rho)\rho \sum_{i=0}^{\infty} i\rho^{i-1}$

$= (1-\rho)\rho(\frac{d}{d\rho} \sum_{i=0}^{\infty} \rho^i)$

$= \frac{\rho}{1-\rho}$

Part 5: Mean hitting time

Consider a discrete-time Markov process $(X_k : k \geq 0)$, with state space $\{1, 2, 3, 4, 5, 6\}$. Suppose the states are arranged in the triangle shown, and given $X_k = i$, the next state X_{k+1} is one of the two neighbors of i, selected with probability 0.5 each. Suppose $P\{X_0 = 1\} = 1$.



(a) Let $\tau_B = \min\{k : X_k \in \{3, 4, 5\}\}$. So τ_B is the time the base of the triangle is first reached. Find $E[\tau_B]$.
(b) Let $\tau_3 = \min\{k : X_k = 3\}$. Find $E[\tau_3]$.
(c) Let τ_C be the first time $k \geq 1$ such that both states 3 and 5 have been visited by time k. Find $E[\tau_C]$.
(d) Let τ_R denote the first time $k \geq \tau_C$ such that $X_k = 1$. That is, τ_R is the first time the process returns to vertex 1 of the triangle after reaching both of the other vertices. Find $E[\tau_R]$.

Solution:

(a) Let h_i denote the mean time to hit $\{3,4,5\}$ from initial state i.

$h_1 = E[\tau_B], h_2 = h_6, h_1 = 1 + h_2, h_2 = 0.5(1 + h_1) + 0.5(1 + h_3), h_3 = 0$

$\Rightarrow h_1 = 4, h_2 = 3$

$\Rightarrow E[\tau_B] = 4$

(b) Let h_i denote the mean time to hit $\{3\}$ from initial state i.

$h_1 = h_5, h_2 = h_4, h_1 = 0.5(1 + h_2) + 0.5(1 + h_6), h_2 = 0.5(1 + h_1) + 0.5(1 + h_3), h_3 = 0$

$\Rightarrow h_1 = 8, h_2 = 5, h_6 = 9$

$\Rightarrow E[\tau_3] = h_1 = 8$

(c) We can use the result from (a) and (b), we know that $3 \to 5$ and $5 \to 3$ are equivalent to $1 \to 3$, therefore $E[\tau_C] = 4 + 8 = 12$.

(d) Use the result from (b) and (c), $E[\tau_R] = 12 + 8 = 20$.